Limit Behavior of Swarms of Coupled Agents

Submitted to the Mathematics Department of
the United States Naval Academy in Partial
Fulfillment of the Requirements for a Degree
in Mathematics with Honors

by Carl Kolon
1 May 2018

Accepted by the Honors Committee

Advisor

G. Nakos

Chairman

First Reader

Prof. Will Traves

Chairman of Mathematics Department

Second Reader

Date

Abstract

Swarms have been studied both in an attempt to understand natural processes and to control systems of autonomous agents. These swarm models are typically based on gradient systems of differential equations. This honors project examines the limit behavior of these systems, and extensions of these systems to Riemannian manifolds.

Acknowledgments

I would like to thank my advisor, Prof. Kostya Medynets, for the immense work that he has devoted to this project. I would also like to thank Prof. Irina Popovici and Prof. Alexis Alevras for their fruitful discussion and attention to this work. Finally, I would like to thank Dr. Ira Schwartz of Naval Research Laboratory for his guidance on the problem of interacting swarms.

1 Introduction

Swarming is a phenomenon observed in almost every type of living being [1, 2, 3]. Many models have been proposed for swarms, both Eulerian models (using continuous dynamics) and Lagrangian models (predicting the behavior of individual agents) [4]. More recently, swarm dynamics have been studied to control robotic agents, especially by the United States military [5, 6]. With this application in mind, we choose to study Lagrangian models as a means to control agents individually. Among Lagrangian models, the most commonly studied swarms are known as potential or gradient swarms: swarms which attempt to minimize a potential function by moving into a desirable configuration [7].

Our work is primarily concerned with the nonlinear parabolic potential model for a swarm. In the model, N agents with position vectors $\mathbf{r}_i$ obey the following equation of motion:

$$\ddot{\mathbf{r}}_i = (1 - \dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i)\dot{\mathbf{r}}_i - \mathbf{r}_i + \frac{1}{N}\sum_{j=1}^{N} \mathbf{r}_j \tag{1}$$

This model has been studied numerically before [8, 9], but to our knowledge there does not yet exist a rigorous mathematical characterization of its limit behavior for any number of agents. We are motivated by a desire to understand the system from a mathematical point of view.

We note that Haraux and Jendoubi [10] investigated second-order systems of the form:

$$\ddot{\mathbf{r}}_i = \mathbf{f}(\dot{\mathbf{r}}_i) - \frac{1}{N}\sum_{j=1}^{N} \nabla_{\mathbf{r}_i} U(\mathbf{r}_i, \mathbf{r}_j),$$

where $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ is a vector-valued function of the agent's velocity and $U : \mathbb{R}^n \to \mathbb{R}$ is a potential function which is smooth and radially unbounded.

If $\mathbf{f}(\dot{\mathbf{r}}_i) \cdot \dot{\mathbf{r}}_i$ is negative-definite, the system will converge to a limit configuration which minimizes U locally. We would like to stress that this negative-definiteness condition does not hold for Equation (1).

Previous numerical experiments have shown that the system either converges to a rotating circular limit cycle with a fixed center of mass, or the agents clump together and move along a straight line. We will show that this is not always the case, and the behavior is sometimes more nuanced (see Section 3.4). Our goal is to investigate stability of the system's circular rotating state. Notice that the system is translation-invariant, and that when the center of mass is fixed, the system decouples into N independent equations.

To understand the decoupled state, we first investigate the behavior of a simpler system, which we call the simplified parabolic potential model. It contains one agent with position vector $\mathbf{r}$, which is attracted to the origin as though it is the center of mass. This yields the following equation of motion:

$$\ddot{\mathbf{r}} = (1 - \dot{\mathbf{r}} \cdot \dot{\mathbf{r}})\dot{\mathbf{r}} - \mathbf{r} \tag{2}$$

We prove the following about the simplified parabolic potential model:

Theorem. *Suppose $\mathbf{r} : \mathbb{R} \to \mathbb{R}^n$ is a solution of Equation (2).*

(i) If $\mathbf{r}$ and $\dot{\mathbf{r}}$ are parallel at some instant, they remain parallel and follow a unique limit cycle (Theorem 4).

(ii) The origin of the phase space is the unique equilibrium point of the system, and it is unstable (Theorem 5).

(iii) If $\mathbf{r}$ and $\dot{\mathbf{r}}$ are not parallel at some instant, then the system converges to a circular limit cycle of radius 1 with speed 1 centered on the origin (Theorem 6).

In the next result, we do local stability analysis for the original system with two agents.

Theorem. *Let $N = 2$. Circular limit cycles satisfying Equation (1) with stationary center of mass are locally stable up to translation (Theorem 7).*

The other focus of our project is an extension of the parabolic potential model to Riemannian manifolds. The goal was to find a swarm model which would exhibit similar behavior to the parabolic model on the manifold: the limit cycles would be circular with respect to the Riemannian metric, the speed of translation would be constant, and the agents would move along geodesics if there were no interaction. As far as we know, this has not been studied before. In Section 4.1, we present such a model for any gradient swarm. Our observations through numerical simulation have shown that the behavior of a swarm in Euclidean space and the same swarm on a manifold tend to be similar.

The structure of the paper follows. In Section 2 we cover some of the theory necessary for our results. In Section 3 we define and characterize some swarms in Euclidean space. In Section 3.3, we characterize the global behavior of a simplified, decoupled system with one agent by constructing an explicit Lyapunov function for the system and using Lasalle's invariance principle. In Section 3.4, we prove that the "typical" limit behavior of a rotating state is locally stable for a system of two agents using Lyapunov's indirect (or linearization) method. In Section 4.1, we use ideas from differential geometry to generate gradient swarms on Riemannian manifolds. We present a swarm model whose limit cycles are precisely the circles (the set of points equidistant from their common center) on the manifold.

2 Lyapunov Stability Theory

Much of our work relies on Lyapunov stability theory. In particular, we use Lasalle's invariance principle to characterize the behavior of the single-agent system, and Lyapunov's indirect (linearization) method to characterize the local behavior of multi-agent systems.

Theorem 1 (Lasalle)**.** *Let an autonomous dynamical system defined on a region $D \subset \mathbb{R}^n$ be given, with the equation of motion:*

$$\dot{\mathbf{x}} = f(\mathbf{x}) \tag{3}$$

Where $f : D \to \mathbb{R}^n$ is a locally Lipschitz map. Let $\Omega \in D$ be a compact set that is positively invariant with respect to (3). Let $V : D \to \mathbb{R}$ be a continuously differentiable function such that $\dot{V}(\mathbf{x}) \leq 0$ for all trajectories in Ω. Let E be the set of all points in Ω where $\dot{V}(\mathbf{x}) = 0$. Let M be the largest invariant set in E. Then every solution starting in Ω approaches M as $t \to \infty$.

Proof. This proof adapted from a proof published in *Nonlinear Systems*, by Khalil [11].

Let $\mathbf{x}(t)$ be a solution of (3) starting in Ω. $\dot{V}(\mathbf{x}) \leq 0$ in Ω, and trajectories beginning in Ω will remain within Ω for all time, since Ω is invariant. Therefore, $V(\mathbf{x}(t))$ is a decreasing function of t. $V(\mathbf{x})$ is continuous on the compact set Ω, so it is bounded from below on Ω. Therefore, $V(\mathbf{x}(t))$ has a limit a as $t \to \infty$. Ω is compact, so it is closed, so the positive limit set L^+ is in Ω. For any $\mathbf{p}$ in L^+, there is a sequence t_n such that $t_n \to \infty$ and $\mathbf{x}(t_n) \to \mathbf{p}$ as $n \to \infty$. By continuity of $V(\mathbf{x})$, $V(\mathbf{p}) = \lim_{n\to\infty} V(\mathbf{x}(t_n)) = a$. Therefore, $V(\mathbf{x}) = a$ on L^+. Thus:

$$L^+ \subset M \subset E \subset \Omega$$

Since $\mathbf{x}(t)$ is bounded, $\mathbf{x}(t)$ approaches (becomes arbitrarily close to a member of) L^+ as $t \to \infty$. Therefore, $\mathbf{x}(t)$ approaches M as $t \to \infty$. □

We also rely on the following theorems about linearized systems:

Theorem 2. *For the linear system $\dot{x} = Ax$, the equilibrium point $x = 0$ is stable if and only if all eigenvalues of A satisfy $\mathrm{Re}(\lambda_i) \leq 0$ and for every eigenvalue with $\mathrm{Re}(\lambda_i) = 0$ and algebraic multiplicity $q_i \geq 2$, $\mathrm{rank}(A - \lambda_i I) = n - q_i$, where n is the dimension of x. The equilibrium point $x = 0$ is (globally) asymptotically stable if and only if all eigenvalues of A satisfy $\mathrm{Re}(\lambda_i) < 0$.*

For a proof, see Khalil [11], Theorem 4.5.

Theorem 3 (Lyapunov's indirect (linearization) method)**.** *Let $x = 0$ be an equilibrium point of the nonlinear system*

$$\dot{x} = f(x)$$

where $f : D \to \mathbb{R}^n$ is continuously differentiable and D is a neighborhood of the origin. Let

$$A = \left. \frac{\partial f}{\partial x}(x) \right|_{x=0}$$

Then:

1. *The origin is asymptotically stable if $\mathrm{Re}(\lambda_i) < 0$ for all eigenvalues of A.*
2. *The origin is unstable if $\mathrm{Re}(\lambda_i) > 0$ for one or more eigenvalues of A.*

For a proof, see Khalil [11], Theorem 4.7.

3 Gradient Swarm Models

The dynamical systems we will study will all be first or second-order autonomous systems. A swarm is a dynamical system of coupled agents. Agents each have a position vector, and some equation of motion which depends on the position of other agents. All swarms we study are homogeneous: all agents have the same equation of motion.

We define an equilibrium configuration of a swarm of agents with positions $\mathbf{r}_i$ as a list of position vectors $(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N)$ such that agents located at these positions will all remain motionless.

For some functions $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ and $U : \mathbb{R}^n \to \mathbb{R}$, Haraux and Jendoubi [10] define first- and second-order gradient systems respectively as follows:

$$\dot{\mathbf{x}} = -\nabla U(\mathbf{x}) \tag{4}$$
$$\ddot{\mathbf{x}} = -\mathbf{f}(\mathbf{x}) - \nabla U(\mathbf{x}) \tag{5}$$

Let a dynamical system of N agents in $\mathbb{R}^n$ be given. A potential function is a function $U : \mathbb{R}^{2n} \to \mathbb{R}$ that assigns a value, or potential, to two agents depending on their position. Examples of commonly-studied potential functions are:

$$U(\mathbf{r}_i, \mathbf{r}_j) = \|\mathbf{r}_i - \mathbf{r}_j\|^2 \qquad \text{(parabolic potential)}$$
$$U(\mathbf{r}_i, \mathbf{r}_j) = \frac{10}{9} \exp\left(-\frac{4\|\mathbf{r}_i - \mathbf{r}_j\|}{3}\right) - \exp(-\|\mathbf{r}_i - \mathbf{r}_j\|) \qquad \text{(Morse potential)}$$

We work with potential functions satisfying several criteria:

1. U is smooth (infinitely differentiable).
2. U is bounded below.
3. All sublevel sets of U are bounded (sublevel sets are sets of the form $\{x : U(x) < k\}$ for some $k \in \mathbb{R}$).

3.1 First-Order

A first-order gradient swarm is a dynamical system of N agents in n dimensions satisfying the following equation of motion:

$$\dot{\mathbf{r}}_i = -\frac{1}{N} \sum_{j=1}^{N} \nabla_{\mathbf{r}_i} U(\mathbf{r}_i, \mathbf{r}_j)$$

Where $\nabla_{\mathbf{r}_i}$ signifies taking the gradient with respect to $\mathbf{r}_i$.

We define the swarm potential, S, as the sum of the potential functions for every pair of agents. Mogilner et al. [4] showed that one-dimensional first-order gradient swarms always converge to an equilibrium configuration if U satisfies the properties of a potential function, and produced a way to prove this in arbitrarily many dimensions. Alternatively, we may view the positions of all agents as one large vector $\mathbf{R}$, and the swarm as a large gradient system of the form:

$$\dot{\mathbf{R}} = -\nabla S(\mathbf{R})$$

It is well-known that systems like this converge to equilibrium points if S is bounded below, and all sublevel sets of S are bounded.

3.2 Second-Order

A second-order gradient swarm is a dynamical system of N agents in n dimensions satisfying the following equation of motion:

$$\ddot{\mathbf{r}}_i = \mathbf{f}(\dot{\mathbf{r}}_i) - \frac{1}{N} \sum_{j=1}^{N} \nabla_{\mathbf{r}_i} U(\mathbf{r}_i, \mathbf{r}_j)$$

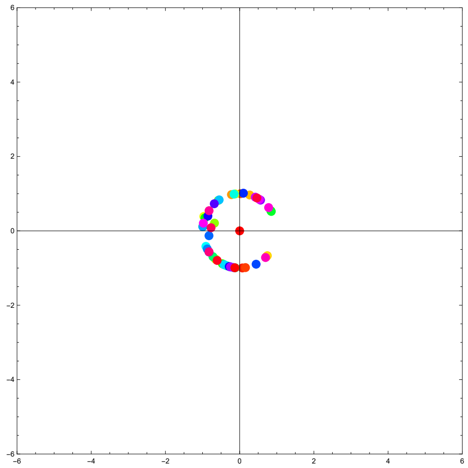


Figure 1: Limit behavior of the parabolic potential model.

Where $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ is a vector-valued function of the agent's velocity.

In the case where $\mathbf{f}(\dot{\mathbf{r}}_i) \cdot \dot{\mathbf{r}}_i$ is negative-definite, a second-order gradient swarm will converge to a limit configuration. Furthermore, the set of limit configurations is equal to the set of limit configurations of the first-order system with the same potential function.

Mogilner et al. [4] argued that this is true because such second-order gradient systems are similar to first-order systems at low speeds [4]. These swarms are high-dimensional cases of the second-order gradient systems studied by Haraux and Jendoubi [10], and therefore will converge to minima of U.

If $\mathbf{f}(\dot{\mathbf{r}}_i) \cdot \dot{\mathbf{r}}_i$ is not negative-definite, second-order gradient swarms do not necessarily converge to equilibrium configurations. An example of a gradient swarm which does not necessarily converge is the nonlinear parabolic potential model:

$$\ddot{\mathbf{r}}_i = (1 - \dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i)\dot{\mathbf{r}}_i - \frac{1}{N}\sum_{j=1}^{N} \nabla_{\mathbf{r}_i} \|\mathbf{r}_i - \mathbf{r}_j\|^2 \tag{6}$$

We have observed through numerical simulation that typically (though not always) the center of mass of the system converges to an equilibrium point, and all agents converge to circular limit cycles around the point (pictured in Figure 6). While the system has been examined (e.g. by Ebeling and Schweitzer [8]), we have never seen proofs of stability concerning this limit behavior.

3.3 Simplified Parabolic Model

In an attempt to understand the swarm, we construct a simplified model with one agent centered on the origin. The system's behavior is given by the following equation of motion:

$$\ddot{\mathbf{r}} = (1 - \dot{\mathbf{r}} \cdot \dot{\mathbf{r}})\dot{\mathbf{r}} - \mathbf{r} \tag{7}$$

Theorem 4 (Original)**.** *If $\mathbf{r}$ and $\dot{\mathbf{r}}$ are parallel at some instant, then they remain parallel and follow a unique limit cycle.*

Proof. First, suppose that $\mathbf{r}$ and $\dot{\mathbf{r}}$ are parallel. I.e., for some unit vector $\hat{\mathbf{r}}$, we have:

$$\begin{aligned} \mathbf{r} &= a\hat{\mathbf{r}} \\ \dot{\mathbf{r}} &= b\hat{\mathbf{r}} \end{aligned}$$

Then (7) yields:

$$\ddot{\mathbf{r}} = (b - b^3 - a)\hat{\mathbf{r}}$$

The acceleration is parallel to $\hat{\mathbf{r}}$, so the entire system is constrained to the line $k\hat{\mathbf{r}}$. We can rewrite differential equations for a and b:

$$\begin{aligned} \dot{a} &= b \\ \dot{b} &= b - b^3 - a \end{aligned} \tag{8}$$

We may rewrite this system as a Lienard system. Use the substitution:

$$\begin{aligned} x &= b \\ y &= -a \\ f(x) &= x^3 - x \end{aligned}$$

Then (8) becomes:

$$\begin{aligned} \dot{x} &= y - f(x) \\ \dot{y} &= -x \end{aligned} \tag{9}$$

Liénard [12] investigated this type of system. This system is a special case studied by Lins, Melo, and Pugh [13]. It has a unique stable limit cycle if:

1. f is continuous,
2. f is odd,
3. f has a unique positive root at $x = k$, and
4. f is monotone increasing for $x > k$.

f satisfies all four properties with $k = 1$, so a unique stable limit cycle exists if $\mathbf{r}$ and $\dot{\mathbf{r}}$ are parallel. □

Theorem 5 (Original)**.** *$a = 0$, $b = 0$ is the unique equilibrium point of the one-dimensional system, and it is unstable.*

Proof. Suppose $\dot{a} = \dot{b} = 0$. Then $b = 0$, and $b - b^3 - a = -a = 0$. So $(0, 0)$ is the unique equilibrium point.

To see that it is unstable, use Lyapunov's indirect method [14]. The Jacobian of the system at $(0, 0)$ is:

$$\mathbf{J} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$$

Which has eigenvalues $\frac{1}{2} \pm \frac{\sqrt{3}}{2}$. Since the real parts of all eigenvalues are positive, the equilibrium point is unstable. □

Theorem 6 (Original)**.** *Suppose that* $\mathbf{r}$ *and* $\dot{\mathbf{r}}$ *are not parallel. Then the system converges to a circular limit cycle of radius 1 with speed 1 centered on the origin.*

Proof. We define scalars u, v and w using the following substitution:

$$\begin{aligned} u &= \mathbf{r} \cdot \mathbf{r} \\ v &= \dot{\mathbf{r}} \cdot \dot{\mathbf{r}} \\ w &= \dot{\mathbf{r}} \cdot \mathbf{r} \end{aligned} \tag{10}$$

Differentiating each scalar, we have:

$$\begin{aligned} \dot{u} &= 2\dot{\mathbf{r}} \cdot \mathbf{r} && = 2w \\ \dot{v} &= 2\ddot{\mathbf{r}} \cdot \dot{\mathbf{r}} = 2(1 - \dot{\mathbf{r}} \cdot \dot{\mathbf{r}})\dot{\mathbf{r}} \cdot \dot{\mathbf{r}} - 2\mathbf{r} \cdot \dot{\mathbf{r}} && = 2v(1-v) - 2w \\ \dot{w} &= \ddot{\mathbf{r}} \cdot \mathbf{r} + \dot{\mathbf{r}} \cdot \dot{\mathbf{r}} = (1 - \dot{\mathbf{r}} \cdot \dot{\mathbf{r}})\dot{\mathbf{r}} \cdot \mathbf{r} - \mathbf{r} \cdot \mathbf{r} + \dot{\mathbf{r}} \cdot \dot{\mathbf{r}} && = w(1-v) - u + v \end{aligned} \tag{11}$$

This system is also subject to the following constraints:

$$\begin{aligned} u &\geq 0 \quad \text{because it is the squared norm of a vector.} \\ v &\geq 0 \quad \text{because it is the squared norm of a vector.} \\ w^2 &\leq uv \quad \text{by the Cauchy-Schwarz Inequality.} \end{aligned} \tag{12}$$

In the case where $w^2 = uv$, we have $(\dot{\mathbf{r}} \cdot \mathbf{r})^2 = (\mathbf{r} \cdot \mathbf{r})(\dot{\mathbf{r}} \cdot \dot{\mathbf{r}})$, so the two vectors are parallel. In this case, the system reduces to the one-dimensional case. If $u = 0$ or $v = 0$, it must be the case that $w^2 = uv$, so we also have the one-dimensional case. If we have $u = v = 0$, we have the unstable equilibrium point in the one-dimensional case. Therefore, we are only interested in the behavior within the following region, which we call Ω:

$$\begin{aligned} u &> 0 \\ v &> 0 \\ w^2 &< uv \end{aligned} \tag{13}$$

Define the Lyapunov function $L : \Omega \to \mathbb{R}$ of the system as:

$$L = u + v - \log(uv - w^2)$$

Since $uv - w^2 > 0$ in Ω, L is defined throughout Ω. It is worth noting that the Lyapunov function does not correspond to any actual physical quantity. It is merely a function that has the properties we require.

L is radially unbounded. I.e. $L \to \infty$ as $|(u, v, w)| \to \infty$. To see this, note that:

$$L = u + v - \log(uv - w^2) \geq (u - \log(u)) + (v - \log(v))$$

Since $u - \log(u)$ and $v - \log(v)$ are unbounded above, L is radially unbounded. Furthermore, L approaches infinity on the boundaries of Ω. For some fixed values of u and v, as $w^2 \to uv$ from below, $\log(uv - w^2) \to -\infty$, so $L \to \infty$. Together, this means that all sublevel sets of L are bounded.

For some constant k, we denote the sublevel set $\{(u, v, w) : L(u, v, w) \leq k\}$ by $\Omega(k)$. Since L is continuous, and all sets $L \leq k$ are closed, all $\Omega(k)$ are closed. Therefore, all $\Omega(k)$ are compact.

L has one stationary point in Ω, at $(1,1,0)$. To show this, take the gradient, and set it equal to zero:

$$\nabla L = \begin{pmatrix} 1 - \frac{v}{uv-w^2} \\ 1 - \frac{u}{uv-w^2} \\ \frac{2w}{uv-w^2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

This yields:

$$\begin{aligned} v &= uv - w^2 & & & (1-u)v &= 0 \\ u &= uv - w^2 & &\Rightarrow & (1-v)u &= 0 \\ w &= 0 & & & w &= 0 \end{aligned} \tag{14}$$

Since u and v are positive in Ω, the only solution is $(u,v,w) = (1,1,0)$.

$(1,1,0)$ is a minimum. To show this, we compute the Hessian of L:

$$\mathbf{H} = \begin{pmatrix} \frac{v^2}{(uv-w^2)^2} & \frac{uv}{(uv-w^2)^2} - \frac{1}{uv-w^2} & -\frac{2vw}{(uv-w^2)^2} \\ \frac{uv}{(uv-w^2)^2} - \frac{1}{uv-w^2} & \frac{u^2}{(uv-w^2)^2} & -\frac{2uw}{(uv-w^2)^2} \\ -\frac{2vw}{(uv-w^2)^2} & -\frac{2uw}{(uv-w^2)^2} & \frac{4w^2}{(uv-w^2)^2} + \frac{2}{uv-w^2} \end{pmatrix}$$

At the point $(1,1,0)$, this becomes:

$$\mathbf{H} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Since this is positive-definite, (1,1,0) is a minimum. Since it is the only stationary point in Ω, and $L \to \infty$ on the boundary of Ω, it is a global minimum.

For all trajectories beginning in Ω:

$$\begin{aligned} \frac{\mathrm{d}L}{\mathrm{d}t} &= \nabla L \cdot \begin{pmatrix} u'(t) \\ v'(t) \\ w'(t) \end{pmatrix} \\ &= \begin{pmatrix} 1 - \frac{v}{uv-w^2} \\ 1 - \frac{u}{uv-w^2} \\ \frac{2w}{uv-w^2} \end{pmatrix} \cdot \begin{pmatrix} 2w \\ 2v(1-v) - 2w \\ w(1-v) - u + v \end{pmatrix} \\ &= \left(2w - \frac{2vw}{uv-w^2}\right) + \left(2v(1-v) - 2w - \frac{2uv(1-v)}{uv-w^2} + \frac{2uw}{uv-w^2}\right) \\ &\quad + \left(\frac{2w^2(1-v)}{uv-w^2} - \frac{2uw}{uv-w^2} + \frac{2vw}{uv-w^2}\right) \\ &= 2v(1-v) - \frac{2(uv-w^2)(1-v)}{uv-w^2} \\ &= -2(1-v)^2 \end{aligned} \tag{15}$$

So L is decreasing on any trajectory beginning in Ω. Therefore, L is indeed a Lyapunov function, and all sublevel sets of L are invariant [14].

We have, for any $\Omega(k)$, $\Omega(k)$ is closed, bounded, and invariant. Given any trajectory $(u(t), v(t), w(t)$, we may apply Theorem 1 on $\Omega(L(u(0), v(0), w(0)))$. Theorem 1 guarantees that (u,v,w) approaches the largest invariant set inside which $\dot{L}(u, v, w) = 0$. All that remains is to show that this set is equal to the point $(1, 1, 0)$.

Since $(1, 1, 0)$ is a global minimum of L, it is in every nonempty sublevel set of L. Suppose that $\dot{L} = 0$ for some trajectory $(u(t), v(t), w(t))$. Then:

$$-2(1 - v(t))^2 = 0 \Rightarrow v(t) = 1 \quad \text{for all } t > 0$$

So by equation (11):

$$\dot{v}(t) = 0 \Rightarrow 2v(t)(1 - v(t)) - 2w(t) = 0 \Rightarrow w(t) = 0 \quad \text{for all } t > 0$$

So:

$$\dot{w}(t) = 0 \Rightarrow w(t)(1 - v(t)) - u(t) + v(t) = 0 \Rightarrow u(t) = 1 \quad \text{for all } t > 0$$

So the only invariant set in any $\Omega(k)$ with $\dot{V}(u(t), v(t), w(t)) = 0$ is the point $(1, 1, 0)$.

Therefore, by Theorem 1, every trajectory beginning in Ω converges to the point $(1, 1, 0)$.

So $\mathbf{r} \cdot \mathbf{r} \to 1$, $\dot{\mathbf{r}} \cdot \dot{\mathbf{r}} \to 1$, and $\dot{\mathbf{r}} \cdot \mathbf{r} \to 0$.

So if $\mathbf{r}(0)$ and $\dot{\mathbf{r}}(0)$ are not parallel, the system converges to a circular limit cycle of radius 1 with speed 1 centered on the origin. □

3.4 Two-Agent Model

We next examine the behavior of two coupled agents, with position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$. The agents have equations of motion:

$$\begin{aligned} \ddot{\mathbf{r}}_1 &= (1 - \dot{\mathbf{r}}_1 \cdot \dot{\mathbf{r}}_1)\dot{\mathbf{r}}_1 - \frac{1}{2}\mathbf{r}_1 + \frac{1}{2}\mathbf{r}_2 \\ \ddot{\mathbf{r}}_2 &= (1 - \dot{\mathbf{r}}_2 \cdot \dot{\mathbf{r}}_2)\dot{\mathbf{r}}_2 - \frac{1}{2}\mathbf{r}_2 + \frac{1}{2}\mathbf{r}_1 \end{aligned} \tag{16}$$

We have not characterized the behavior of the two-agent system globally. There exist at least three behaviors which we have seen in simulation: a one-dimensional oscillation, a rotating behavior, and a translating behavior, pictured in Figure 2.

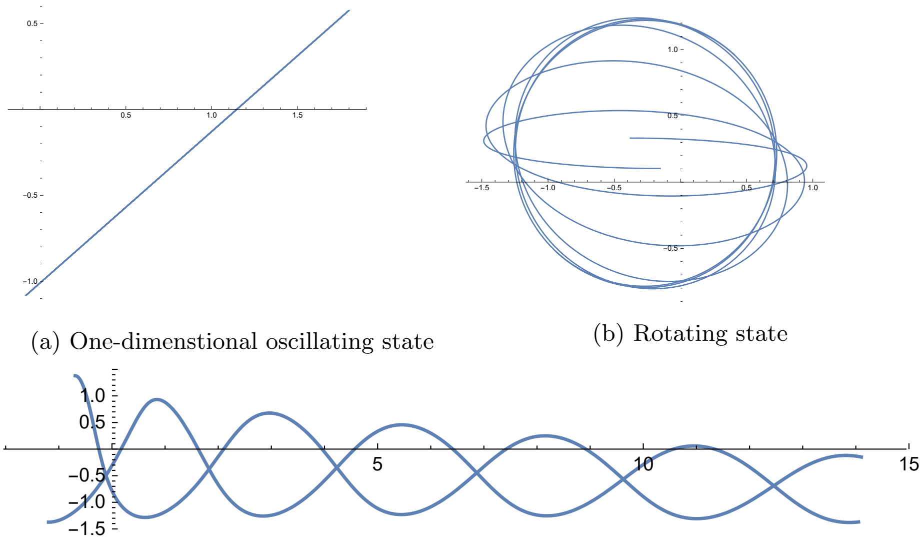
We have shown that the rotating behavior is locally stable by rewriting the system and using Lyapunov's indirect (linearization) method.

Theorem 7 (Original)**.** *Rotating behavior is locally stable for a parabolic potential model of two agents.*

Proof. We begin by reformulating the system using the following substitution:

$$\begin{aligned} u &= |\mathbf{r}_1 - \mathbf{r}_2| \\ v_1 &= |\dot{\mathbf{r}}_1| \\ v_2 &= |\dot{\mathbf{r}}_2| \\ \theta_1 &= \text{angle between } \dot{\mathbf{r}}_1 \text{ and } (\mathbf{r}_1 - \mathbf{r}_2) \\ \theta_2 &= \text{angle between } \dot{\mathbf{r}}_2 \text{ and } (\mathbf{r}_2 - \mathbf{r}_1) \end{aligned} \tag{17}$$

We now differentiate each term.



(a) One-dimenstional oscillating state

(b) Rotating state

(c) Translating state

Figure 2: Different behaviors of a swarm of two agents. Different initial conditions produce these behaviors. The plotted lines are the trajectories of two agents on the plane, with varying initial conditions.

$$\begin{aligned}\dot{u} &= \frac{1}{|\mathbf{r}_1 - \mathbf{r}_2|}(\dot{\mathbf{r}}_1 + \dot{\mathbf{r}}_2) \cdot (\mathbf{r}_2 - \mathbf{r}_1) \\ &= \frac{1}{u}(uv_1 \cos\theta_1 + uv_2 \cos\theta_2) \\ &= v_1 \cos\theta_1 + v_2 \cos\theta_2\end{aligned} \tag{18}$$

$$\begin{aligned}\dot{v}_1 &= \frac{1}{|\dot{\mathbf{r}}_1|}(\ddot{\mathbf{r}}_1 \cdot \dot{\mathbf{r}}_1) \\ &= \frac{1}{v_1}((1 - \dot{\mathbf{r}}_1 \cdot \dot{\mathbf{r}}_1)\dot{\mathbf{r}}_1 \cdot \dot{\mathbf{r}}_1 - \frac{1}{2}(\mathbf{r}_1 - \mathbf{r}_2) \cdot \dot{\mathbf{r}}_1) \\ &= \frac{1}{v_1}((1 - v_1^2)v_1^2 - \frac{1}{2}uv_1 \cos\theta_1) \\ &= (1 - v_1^2)v_1 - \frac{1}{2}v_1 \cos\theta_1\end{aligned} \tag{19}$$

A symmetrical argument can be used to show that:

$$\dot{v}_2 = (1 - v_2^2)v_2 - \frac{1}{2}v_2 \cos\theta_2 \tag{20}$$

The final two derivatives can be calculated implicitly. First, denote the angle between $\dot{\mathbf{r}}_1$ and $\dot{\mathbf{r}}_2$ as ϕ. We can see from Figure 3 that $\phi = \pi - \theta_1 - \theta_2$. Then we have:

$$\cos\phi = \cos(\pi - \theta_1 - \theta_2) = -\cos(\theta_1 + \theta_2) = \sin\theta_1 \sin\theta_2 - \cos\theta_1 \cos\theta_2 \tag{21}$$

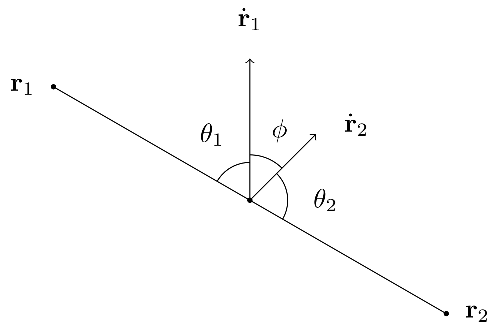


Figure 3: Calculating the angle between $\dot{\mathbf{r}}_1$ and $\dot{\mathbf{r}}_2$.

We can now calculate the final two derivatives implicitly. First, note that:

$$uv_1 \cos\theta_1 = \dot{\mathbf{r}}_1 \cdot (\mathbf{r}_1 - \mathbf{r}_2) \tag{22}$$

Differentiating the left hand side yields:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(uv_1 \cos\theta_1) &= (\dot{u}v_1 + u\dot{v}_1)\cos\theta_1 - uv_1\dot{\theta}_1 \sin\theta_1 \\ &= v_1^2 \cos^2\theta_1 + v_1 v_2 \cos\theta_1 \cos\theta_2 + uv_1(1 - v_1^2)\cos\theta_1 - \frac{1}{2}u^2 \cos^2\theta_1 - uv_1\dot{\theta}_1 \sin\theta_1 \end{aligned} \tag{23}$$

Differentiating the right hand side:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(\dot{\mathbf{r}}_1 \cdot (\mathbf{r}_1 - \mathbf{r}_2)) &= \ddot{\mathbf{r}}_1 \cdot (\mathbf{r}_1 - \mathbf{r}_2) + \mathbf{r}_1 \cdot \mathbf{r}_1 - \mathbf{r}_1 \cdot \mathbf{r}_2 \\ &= (1 - \dot{\mathbf{r}}_1 \cdot \dot{\mathbf{r}}_1)\dot{\mathbf{r}}_1 \cdot (\mathbf{r}_1 - \mathbf{r}_2) - \frac{1}{2}(\mathbf{r}_1 - \mathbf{r}_2)\cdot(\mathbf{r}_1 - \mathbf{r}_2) + \mathbf{r}_1 \cdot \mathbf{r}_1 - \mathbf{r}_1 \cdot \mathbf{r}_2 \\ &= (1 - v_1^2)uv_1 \cos\theta_1 - \frac{1}{2}u^2 + v_1^2 + v_1 v_2 \cos\theta_1 \cos\theta_2 - v_1 v_2 \sin\theta_1 \sin\theta_2 \end{aligned} \tag{24}$$

Equating (23) and (24) and canceling:

$$v_1^2 \cos^2\theta_1 - \frac{1}{2}u^2 \cos^2\theta_1 - uv_1\dot{\theta}_1 \sin\theta_1 = -\frac{1}{2}u^2 + v_1^2 - v_1 v_2 \sin\theta_1 \sin\theta_2 \tag{25}$$

Rearranging for $\dot{\theta}_1$:

$$\begin{aligned} \dot{\theta}_1 &= \frac{\frac{1}{2}u^2 - \frac{1}{2}u^2\cos^2\theta_1 - v_1^2 + v_1^2\cos^2\theta_1 + v_1 v_2 \sin\theta_1 \sin\theta_2}{uv_1 \sin\theta_1} \\ &= \frac{\frac{1}{2}u^2 \sin^2\theta_1 - v_1^2 \sin^2\theta_1 + v_1 v_2 \sin\theta_1 \sin\theta_2}{uv_1 \sin\theta_1} \\ &= \left(\frac{u}{2v_1} - \frac{v_1}{u}\right)\sin\theta_1 + \frac{v_2}{u}\sin\theta_2 \end{aligned} \tag{26}$$

A symmetrical argument shows that:

$$\dot{\theta}_2 = \left(\frac{u}{2v_2} - \frac{v_2}{u}\right)\sin\theta_2 + \frac{v_1}{u}\sin\theta_1 \tag{27}$$

Equations (18), (19), (20), (26), and (27) give the following system of differential equations:

$$
\begin{aligned}
\dot{u} &= v_1 \cos\theta_1 + v_2 \cos\theta_2 \\
\dot{v}_1 &= (1 - v_1^2)v_1 - \frac{1}{2} v_1 \cos\theta_1 \\
\dot{v}_2 &= (1 - v_2^2)v_2 - \frac{1}{2} v_2 \cos\theta_1 \\
\dot{\theta}_1 &= \left(\frac{u}{2v_1} - \frac{v_1}{u}\right) \sin\theta_1 + \frac{v_2}{u} \sin\theta_2 \\
\dot{\theta}_2 &= \left(\frac{u}{2v_2} - \frac{v_2}{u}\right) \sin\theta_2 + \frac{v_1}{u} \sin\theta_1
\end{aligned}
\tag{28}
$$

The rotating behavior corresponds to the case when the distance between particles and the center of mass is 1, and therefore $u = 2$. The speed of both agents v_1 and v_2 will be 1. We also expect that the direction of rotation will be orthogonal to the vector between the two agents, and in opposite directions, giving us that θ_1 and θ_2 are both $\frac{\pi}{2}$ or both $-\frac{\pi}{2}$. This gives us the following two points: $(2, 1, 1, \pi/2, \pi/2)$, and $(2, 1, 1, -\pi/2, -\pi/2)$. At these two points, we linearize the system by calculating the Jacobian matrices:

$$
\begin{aligned}
\mathbf{J}(2, 1, 1, \pi/2, \pi/2) &= \begin{pmatrix} 0 & 0 & 0 & -1 & -1 \\ 0 & -2 & 0 & 1 & 0 \\ 0 & 0 & -2 & 0 & 1 \\ \frac{1}{2} & -\frac{3}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & -\frac{3}{2} & 0 & 0 \end{pmatrix} \\
\mathbf{J}(2, 1, 1, -\pi/2, -\pi/2) &= \begin{pmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & -2 & 0 & -1 & 0 \\ 0 & 0 & -2 & 0 & -1 \\ -\frac{1}{2} & \frac{3}{2} & -\frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & \frac{3}{2} & 0 & 0 \end{pmatrix}
\end{aligned}
\tag{29}
$$

We calculated the values of the eigenvalues of these matrices. For both matrices, the eigenvalues are $(-1.544, -1 \pm i, -0.228 \pm 1.115i)$. Since all these eigenvalues are in the negative complex half-plane, local stability follows from Lyapunov's indirect method [14]. □

3.5 Systems with More Agents

We are currently investigating stability of systems with more than two agents. For a system of N agents, we use the following change of coordinates to describe the position of the ith agent in system (1):

$$
\begin{aligned}
u_i &= \text{ Distance between the } i\text{th agent and the center of mass} \\
v_i &= \text{ Speed of the } i\text{th agent} \\
\theta_i &= \text{ Angle between the } i\text{th agent and the positive } x\text{-axis} \\
\phi_i &= \text{ Angle between the vector } (r_i - R) \text{ and the vector } \dot{r}_i
\end{aligned}
\tag{30}
$$

Where r_i is the position vector of the ith agent, and R is the center of mass of the system. This leads to the following system of differential equations:

$$\begin{aligned}
\dot{u}_i &= v_i \cos\phi_i - \frac{1}{N}\sum_{j=1}^{N} v_j \cos(\phi_j + \theta_j - \theta_i) \\
\dot{v}_i &= (1 - v_i^2)v_i - u_i \cos\phi_i \\
\dot{\theta}_i &= \frac{v_i}{u_i}\sin\phi_i - \frac{1}{u_i N}\sum_{j=1}^{N} v_j \sin(\phi_j + \theta_j - \theta_i) \\
\dot{\phi}_i &= (\frac{u_i}{v_i} - \frac{v_i}{u_i})\sin\phi_i + \frac{1}{u_i N}\sum_{j=1}^{N} v_j \sin(\phi_j + \theta_j - \theta_i)
\end{aligned} \tag{31}$$

This system shifts the center of mass onto the origin. It is therefore necessary to include the constraint that $\sum_{i=1}^{N}(u_i\cos\theta_i, u_i\sin\theta_i) = (0,0)$. As might be expected, the manifold created by this constraint is invariant, so we may restrict ourselves to studying the behavior of the system on this manifold.

We believe that the following conjecture is true:

Conjecture 1. *For any integer $N > 0$, the system (31) of N agents has a stable manifold at $\left(u_i = 1, v_i = 1, \phi_i = \pm\frac{\pi}{2}, \sum_{i=1}^{N}(u_i\cos\theta_i, u_i\sin\theta_i) = (0,0)\right)$, given initial conditions on the invariant manifold $\sum_{i=1}^{N}(u_i\cos\theta_i, u_i\sin\theta_i) = (0,0)$.*

This would imply that the ring state of the system is locally stable, because in a neighborhood of the ring state, the speed of each agent converges to 1, the velocity converges to 1, and the velocity is orthogonal to the speed.

We have shown this using a linearization for 3 agents, but with higher numbers of agents, the system has dimensions of neutral stability. For example, there are many possible configurations in a swarm of five agents: we may perturb all agents along the unit circle in a way which maintains the center of mass. This type of perturbation is pictured in Figure 4.

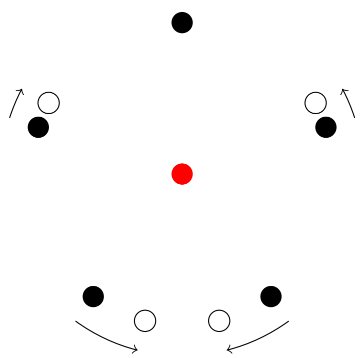


Figure 4: A perturbation which keeps the center of mass constant in a five-agent system.

MIDN Carl Kolon, Prof. Kostya Medynets, and Prof. Irina Popovici, all at USNA, are currently attempting to use center manifold analysis to prove that this conjecture is indeed true for any number of agents.

4 Motion on Riemannian Manifolds

We have generalized the concept of a gradient swarm to Riemannian manifolds. To our knowledge, this has never been done before. So far, we have only observed the behavior of the system numerically, but the behavior of gradient swarms on manifolds appears to mimic their behavior in Euclidean space. As is common in differential geometry, we use Einstein summation notation for our equations.

A Riemannian manifold is a smooth manifold M equipped with an inner product $\langle \cdot, \cdot \rangle_p$ (the metric tensor) on the tangent space T_pM at every point p. The inner product varies smoothly along any vector field on M. On a chart of a Riemannian manifold with coordinates x_i, the metric tensor is often represented as g, such that for two tangent vectors $\mathbf{u}$ and $\mathbf{v}$, $\langle \mathbf{u}, \mathbf{v} \rangle = g_{ij}u^iv^j$.

A geodesic is a locally length-minimizing curve. A geodesic γ satisfies the geodesic equation [15]:

$$\ddot{\gamma}^a + \Gamma^a_{ij}\dot{\gamma}^i\dot{\gamma}^j = 0$$

in all coordinate directions a, where Γ is the Christoffel symbol of the second kind, defined as:

$$\Gamma^a_{ij} = \frac{1}{2}g^{ak}(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k})$$

An agent on the manifold with position vector $\mathbf{x} = (x^1, x^2, \ldots, x^a)$ moving with no external acceleration will follow geodesic paths, so such an agent will satisfy the following equation of motion:

$$\ddot{x}^a + \Gamma^a_{ij}\dot{x}^i\dot{x}^j = 0$$

The concept of the gradient is different on a Riemannian manifold as well. For a function $f : M \to \mathbb{R}$, and some vector field V on M, we desire that the gradient have the following property:

$$(\nabla f)^T gV = df^T(V)$$

Where df is the directional derivative of f, and T represents the transpose. Since this is an identity for any V:

$$(\nabla f)^T g = df^T$$

Which gives:

$$g^T \nabla f = df$$

And since inner products are symmetric:

$$\nabla f = g^{-1} df$$

This gives the following form for the gradient on a Riemannian manifold:

$$\nabla f = \sum_k g^{ik} \frac{\partial f}{\partial x^k} \mathbf{e}_i \tag{32}$$

4.1 Swarming on Riemannian Manifolds

The original swarm in Euclidean space has equation of motion:

$$\ddot{\mathbf{r}}_i = \mathbf{f}(\dot{\mathbf{r}}) - \frac{1}{N}\sum_{j=1}^{N} \nabla_{\mathbf{r}_i} U(\mathbf{r}_i, \mathbf{r}_j) \tag{33}$$

To make this swarm suitable for manifolds, we must do two things:

1. We must change the flat representation of the gradient to the Riemannian manifold representation.
2. We must ensure that, if $\mathbf{f}$ and U are everywhere zero, the agent moves along a geodesic.

To do this, we define a gradient swarm of N agents with positions $\mathbf{r}_i = (r_i^1, \ldots, r_i^n)$ (in curvilinear coordinates) by the following equation of motion:

$$\ddot{r}_i^a = f^a(\dot{\mathbf{r}}_i) - \Gamma^a_{jk}\dot{r}_i^j \dot{r}_i^k - \frac{1}{N}\sum_{j=1}^{N}\sum_{k=1}^{n} g^{ak}\frac{\partial}{\partial r_i^k} U(\mathbf{r}_i, \mathbf{r}_j) \tag{34}$$

Our numerical simulations of this system show that it inherits many familiar properties of gradient swarms. To replicate the parabolic potential model, we used the following values of $\mathbf{f}$ and U:

$$\begin{aligned} f^a(\dot{\mathbf{r}}) &= (1 - g_{ij}\dot{r}^i\dot{r}^j)r^a \\ U(\mathbf{r}_i, \mathbf{r}_j) &= \text{the squared length of the shortest path between } \mathbf{r}_i \text{ and } \mathbf{r}_j \end{aligned} \tag{35}$$

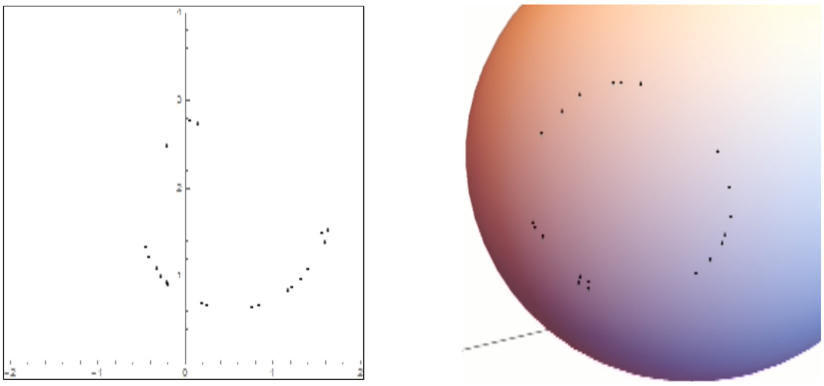
On some Riemannian manifolds (e.g. the sphere and hyperbolic half-plane), there are closed-form expressions for both $\mathbf{f}$ and U. We numerically simulated the parabolic potential model on these surfaces, and found that the swarms still formed circular limit cycles (see Figure 5). Our current methods for simulating the system when there are not closed-form expressions for $\mathbf{f}$ and U are too computationally intensive to use, but we are working on methods which use less computing power.

4.2 Modifying Limit Cycles

We used the concepts developed with the generalization of gradient swarms to modify the limit cycles of the nonlinear parabolic potential model. Other methods to achieve this were considered by Medynets and Schwartz [9]. By using a constant (flat) matrix as our metric tensor, we can skew the motion of the agents. In this case, there is also a closed form for the distance between agents. This allows us to numerically simulate the modified swarms, which have oval limit cycles rather than circles.

5 Technical Implementation

We are interested in the applications of this research to the motion of swarms of unmanned vehicles. We have implemented swarm models like the nonlinear



(a) Swarm in hyperbolic half-plane (b) Swarm on sphere

Figure 5: Generalized parabolic potential swarms simulated on two different manifolds.

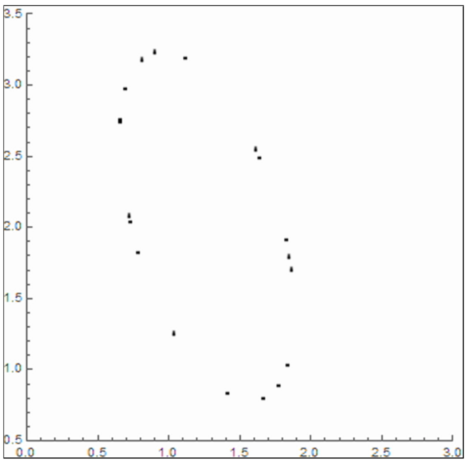


Figure 6: Modifying the parabolic potential model to produce a swarm with oval limit cycles.

parabolic potential model in Wolfram Mathematica. We then designed a package in ROS, the Robot Operating System, to communicate the desired velocity (generated by numerically solving the differential equation in Mathematica) to UAV models in Gazebo, a high-fidelity physics simulator. Figure 7 shows the simulated robotic swarm in action. We hope to extend our simulations in Gazebo to demonstrate our results on Riemannian manifolds.

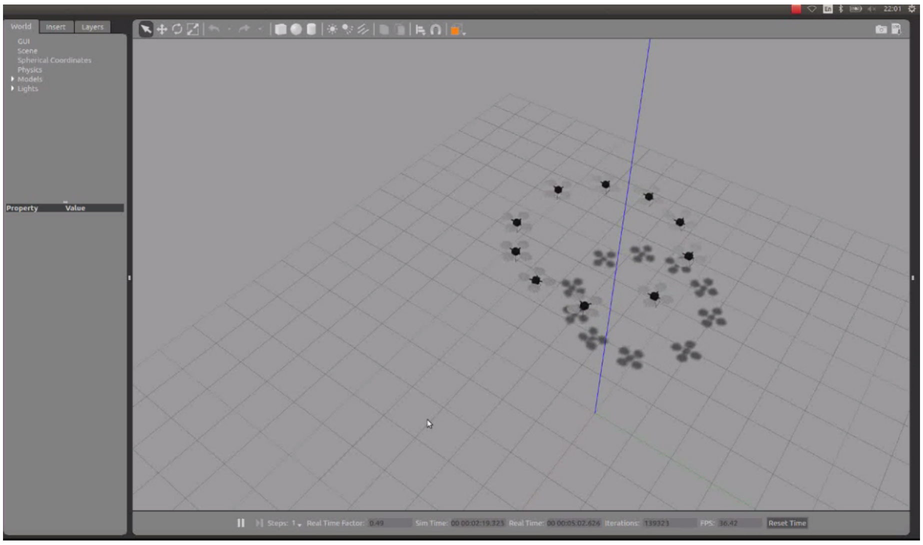


Figure 7: Quadrotors following the limit cycle of the nonlinear parabolic potential swarm in Gazebo. The agents are controlled by ROS, which is feeds velocity and position data between Gazebo and Mathematica. Note the similarity to the limit behavior pictured in Figure 1.

References

[1] M. Ciszak et al. "Swarming Behavior in Plant Roots". In: *PLoS ONE* 7 (2012), p. 1. DOI: 10.1371/journal.pone.0029759.

[2] D. Kearns. "A field guide to bacterial swarming motility". In: *Nature Reviews Microbiology* 8 (2010), pp. 634–644.

[3] C. Reynolds. "Flocks, herds and schools: A distributed behavioral model." In: *Computer Graphics* 21.(4) (1987), pp. 25–34.

[4] A. Mogilner et al. "Mutual interactions, potentials, and individual distance in a social aggregation". In: *Journal of Mathematical Biology* 47 (2003), pp. 353–389.

[5] Office of Naval Research. *LOCUST, a Low Cost UAV Swarm*. 2015.

[6] *Unmanned Systems Integrated Roadmap*. Department of Defense. 2013.

[7] M. R. D'Orsogna et al. "Self-Propelled Particles with Soft-Core Interactions: Patterns, Stability, and Collapse". In: *Phys. Rev. Lett.* 96 (10 Mar. 2006), p. 104302. DOI: 10.1103/PhysRevLett.96.104302.

[8] W. Ebeling and F. Schweitzer. "Swarms of Particle Agents with Harmonic Interactions". In: *Theory in Biosciences* 120/3-4 (2001), pp. 207–224.

[9] K. Medynets and I. Schwartz. "Modelling Swarm Convergence with Arbitrarily Shaped Limit Cycles". In preparation. 2016.

[10] A. Haraux and M. A. Jendoubi. "Convergence of Solutions of Second-Order Gradient-Like Systems with Analytic Nonlinearities". In: *Journal of Differential Equations* 144.DE973393 (1998), pp. 313–320.

[11] H.K. Khalil. *Nonlinear Systems*. Pearson Education. Prentice Hall, 2002. ISBN: 9780130673893.

[12] A. Liénard. *Etude des oscillations entretenues*. Revue générale de l'électricité, 1928.

[13] A. Lins, W. de Melo, and C. C. Pugh. "On Liénard's equation". In: *Lecture Notes in Mathematics*. Springer Berlin Heidelberg, 1977, pp. 335–357. DOI: 10.1007/bfb0085364.

[14] A. M. Lyapunov. "The general problem of the stability of motion (originally published in Russian in 1893)". In: *International Journal of Control* 55.3 (Mar. 1992), pp. 531–534. DOI: 10.1080/00207179208934253.

[15] S. Kobayashi and K. Nomizu. *Foundations of Differential Geometry*. A Wiley Publication in Applied Statistics v. 1. Wiley, 1996. ISBN: 9780471157335.